

# THE F4 ALGORITHM FOR EUCLIDEAN RINGS

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**ABSTRACT.** This short note is the generalization of Faugère F4-algorithm for polynomial rings with coefficients in Euclidean rings. This algorithm computes successively a Gröbner basis replacing the reduction of one single s-polynomial in Buchberger's algorithm by the simultaneous reduction of several polynomials.

The concept of Gröbner Bases was introduced by Bruno Buchberger (1965) in the context of his work on performing algorithmic computations in residue classes of polynomial rings. Buchberger's algorithm for computing Gröbner Bases is a powerful tool for solving many important problems in polynomial ideal theory. To compute Gröbner bases efficiently there are many methods like FGLM, Gröbner walk (cf. [2]). Faugère [5] introduced a new efficient method F4 algorithm to compute Gröbner bases using linear algebra, and a selection strategy among the critical pairs which occurs in the computation of Gröbner basis. This algorithm computes successively a Gröbner basis replacing the reduction of one single s-polynomial in Buchberger's algorithm by the simultaneous reduction of several polynomials. In the book of Adams and Loustaunau (cf. [7]) the concept of Gröbner bases over polynomial rings with coefficients in a ring is developed. The aim of this short note is to show that Faugère's F4-algorithm works also in polynomial rings with coefficients in Euclidean rings.

Let  $>$  be a fixed global monomial ordering and  $R$  be an Euclidean ring.

**Definition 1.** We fix the following notations, writing  $f \in R[x_1, \dots, x_n]$ ,  $f \neq 0$ , in a unique way as a sum of non-zero terms

$$f = a_{\alpha_1}x^{\alpha_1} + a_{\alpha_2}x^{\alpha_2} + \dots + a_{\alpha_s}x^{\alpha_s}, \quad x^{\alpha_1} > x^{\alpha_2} > \dots > x^{\alpha_s},$$

and  $a_{\alpha_1}, a_{\alpha_2}, \dots, a_{\alpha_s} \in R$ . We call:

1.  $LM(f) := x^{\alpha_1}$ , the leading monomial of  $f$ ,
2.  $LE(f) := \alpha_1$ , the leading exponent of  $f$ ,
3.  $LT(f) := a_{\alpha_1}x^{\alpha_1}$ , the leading term of  $f$ ,
4.  $LC(f) := a_{\alpha_1}$ , the leading coefficient of  $f$ ,
5. We define the leading monomial and the leading term of 0 to be 0, and 0 to be smaller than any monomial.

6. Let  $G \subset R[x_1, \dots, x_n]$ , then  $L(G) := \langle \{LT(g) \mid g \in G\} \rangle_{R[x_1, \dots, x_n]}$  is the leading ideal of  $G$ .

**Definition 2.** Let  $I \subset R[x_1, \dots, x_n]$  be an ideal. A finite set  $G \subset R[x_1, \dots, x_n]$  is called a Gröbner basis of  $I$  with respect to  $>$  if  $G \subset I$ , and  $L(I) = L(G)$ .

**Definition 3.** Let  $H \subset R[x_1, \dots, x_n]$  be finite set.  $H$  is called interreduced if for all  $p \neq q$ ,  $p, q \in H$ ,  $LT(p)$  does not divide  $LT(q)$ . Furthermore if  $LM(p) \mid LM(q)$  then the remainder  $LC(q) \bmod LC(p)$  of  $LC(q)$  with respect to the division of  $LC(q)$  by  $LC(p)$  in the Euclidean ring is  $LC(q)$ .

The existence of a set of interreduced generators of  $\langle H \rangle$  is given by the following algorithm.

**Algorithm 1.** *Interreduce( $H$ )*

Input :  $H$  a set of polynomials.

Output :  $L$  a set of interreduced polynomials such that  $\langle H \rangle = \langle L \rangle$ .

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todo = 1
• while ( there exist  $h, k \in H, h \neq k, LM(k) \mid LM(h)$  and  $todo = 1$ )
    todo = 0
    • if  $(LT(k) \mid LT(h))$ 
         $h := h - \frac{LT(h)}{LT(k)}k$ ;
        todo = 1
        if  $(h = 0)$ 
             $H := H \setminus \{h\}$ ;
    • else
        if  $(LC(k) \bmod LC(h)) \neq 0$ ;
            compute  $c = \gcd(LC(h), LC(k)) = aLC(h) + bLC(k)$ ;
            if  $(a \text{ is a unit})$ 
                 $h := ah + b\frac{LM(h)}{LM(k)}k$  ;
                todo = 1
            if  $(LC(h) \bmod LC(k) \neq LC(h))$ ;
                 $h := h + \frac{LC(h) \bmod LC(k) - LC(h)}{LC(k)} \cdot \frac{LM(h)}{LM(k)} \cdot k$ ;
                todo = 1
    • return( $H$ );
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**Algorithm 2.** *F4*

Input :  $G$  set of polynomials,  $S$  a selection strategy<sup>1</sup>.

Output : Gröbner basis for  $\langle G \rangle$ .

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<sup>1</sup>A selection strategy  $S$  associates to the pair set  $P$  a subset  $S(P) \subset P$ . A trivial example is  $S(P) = P$ . In the case that  $G$  is a set of homogeneous polynomial  $S(P) = \{(f, g) \in P \mid \deg(\text{lcm}(LM(f), LM(g))) \text{ is minimal}\}$  is a good choice.

- $G = \text{Interreduce}(G);$
- $P := \{(f, g) \mid f, g \in G\};$
- $\text{while}(P \neq \emptyset)$ 
  - $H := \{\frac{cx^\alpha}{LT(g)} \cdot g, \frac{cx^\alpha}{LT(f)} \cdot f \mid (f, g) \in S(P), cx^\alpha = \text{lcm}(LT(f), LT(g))\};$
  - $P := P \setminus S(P);$
  - $H := \text{Interreduce}(H \cup G);$
  - $P := P \cup \{(f, h) \mid f \in G, h \in H, LT(h) \notin \langle LT(g) \mid g \in G \rangle\};$
  - $G := H;$
- $\text{return } (G);$

**Proposition 4.** *The algorithm F4 terminates and the result is a Gröbner basis of the ideal generated by input.*

*Proof.* The termination is a consequence of the ring  $R[x_1, \dots, x_n]$  being noetherian. Each time the set  $G$  is enlarged the leading ideal  $L(G)$  is enlarged properly. This has to stop after finitely many steps.

The result  $G$  is a Gröbner basis of the ideal generated by the input because it satisfies Buchberger's criterion (cf. [3]). Namely the normal form of the  $s$ -polynomial of any two elements of  $G$  with respect to  $G$  is zero. This holds because for  $f, g \in G$ ,  $E = \frac{cx^\alpha}{LT(f)} \cdot f$  and  $F = \frac{cx^\alpha}{LT(g)} \cdot g$  with  $cx^\alpha = \text{lcm}(LT(f), LT(g))$  have been interreduced with elements from  $G$  and  $\text{spoly}(f, g) = E - F$ .  $\square$

**Example 5.** *Let  $R = \mathbb{Z}$  and consider the ideal*

$$I = \langle 2abcd - 2, abc + 2abd + acd + bcd, ab + bc + ad + cd, a + b + c + d \rangle$$

*in  $\mathbb{Z}[a, b, c, d]$  and  $>$  the degrevlex ordering.*

*In a test-implementation of the F4 and Buchberger's algorithm in SINGULAR, in F4, 91 additions of polynomials are needed while the Buchberger's algorithm needs 375 additions.*

*And we get  $G = \{-2, a + b + c + d, -b^2 + d^2, -bc^2 + bcd + c^2d, bcd^2 + cd^3, -cd^4\}$  a Gröbner basis of  $I$ .*

**Remark 6.** *We presented here the idea of Faugère algorithm in its simplest form to make the principle understandable. For an implementation one should keep the pair set as small as possible using Buchberger's criterion (chain criterion, product criterion, (cf. [3])).*

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